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CR55102

by

W. S. Ament 28 feb. 1963 28p refs

OTS PRICE

XEROX $ 2.60 ph

MICROFILM $ 1.04 mf

Scientific Report No. 4
on
(NASA Contract NAS5-585)

Prepared for

National Aeronautics and Space Administration
Goddard Space Flight Center
Greenbelt, Md.

(NASA CR-55102; ~~Report No.~~ NAS-585-7)
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ABSTRACT

This report deals with some general properties of the dyadic Green's function $\underset{\sim}{G}$ for oscillatory electromagnetic waves in a homogeneous anisotropic medium, and with aspects of the theory of antennas in such a medium. The results, byproducts of a calculation of the impedance of a cylindrical dipole antenna in the magnetosphere, are largely independent of the main argument toward that calculation and are believed to have significance for more general problems.

We first give the Fourier representation of $\underset{\sim}{G}(\underset{\sim}{x})$, a triple integral representing a superposition of plane waves of various propagation vectors $\underset{\sim}{k}$. With this representation we show that $\underset{\sim}{G}$ is an even function of its spatial argument $\underset{\sim}{x}$, and that when the representation is written in terms of spherical coordinates in k-space, the radial integral, over a positive R-axis, can be extended to the full R-axis, a form mathematically convenient in applications. From the Fourier representation we also argue that a closed-form expression for $\underset{\sim}{G}$ is generally unobtainable: For a closed form would amount to a closed-form prescription of the surfaces of constant phase, describing the field at great distances from a point source in the anisotropic medium; the constant-phase surfaces are envelopes of families of plane wave-fronts of the Fourier superposition, and the corresponding envelope problem, as found in the 19th century literature, is generally insoluble in closed form.

We then consider the complex power and the variational prescriptions for calculating the input impedance of perfectly conducting antennas fed at a thin slit. For the anisotropic medium the latter prescription loses its variational efficacy to the extent that $\underset{\sim}{G}$, as a dyad, is non-symmetric, as is the case for

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the magnetosphere. When all currents in the antenna can be assumed mutually parallel, the antisymmetric part of $\underset{\sim}{G}$ has no effect and the variational form is again powerful for antenna-current and impedance estimates.

In calculating the impedance of thin cylindrical antennas, a commonly made assumption is that the radiating currents can be taken as concentrated on the cylinder's axis rather than distributed uniformly around its circumference. Via the Fourier representation for $\underset{\sim}{G}$ we are able to prescribe exact corrections for this "axial current" assumption, and to find that it creates no error in the first two, dominant, terms of the usual expansion of the impedance in functions of the cylinder's radius.

AUTHOR

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1. INTRODUCTION AND OUTLINE

We have recently reported a numerical calculation of the input impedance of a thin center-fed cylindrical dipole in a homogeneous magnetospheric medium [1]*, using a previously established [2] generalization of the "complex power" formulation [3]. The numerical results reduce to known ones in the lossless isotropic limit and otherwise appear satisfactory, particularly where a sinusoidal current distribution is properly assumed to hold along each half of the dipole. During the course of the necessary mathematical analysis, we discovered some general results applicable to this and other antenna problems, the most important being a means of casting the impedance calculation into a variational form without substantial changes in the calculational procedures necessary for obtaining numerical estimates. In this report we present the most important of these developments, reserving their application to the numerical impedance estimate for a later report.

In order to establish motivation and underlying ideas, we review briefly some standard electromagnetic and antenna theory. Our first conclusion is that the nonsymmetric dyadic Green's function for a homogeneous anisotropic medium cannot be expressed in closed form, but must be represented, say, through a Fourier integral. Then we review the complex power and the variational [4] integral expressions for the impedance of a perfectly conducting antenna fed at a gap, and discuss their extensions to the anisotropic case and to special antenna geometries. Here we conclude that the variational form loses its

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* Numbers in brackets refer to the corresponding numbers in the References on p. 23.

variational character and consequent efficacy in establishing a "best" antenna current distribution <u>except</u> for geometries in which one may assume that all current elements are mutually parallel. This assumption is proper for a thin dipole. The further "axial current" assumption, that radiating currents in the dipole can be considered as concentrated on the dipole's axis, is standard (and was made in our numerical calculation). The fact that we are forced, for the anisotropic case, to use a Green's function representation, rather than the closed form of the usual isotropic calculation, now leads easily to exact corrections for any errors made by using the foregoing axial current approximation. Some of the mathematical details, and a discussion of the physical problem of antenna impedance in a space-probing antenna, are relegated to appendices.

2. THE GREEN'S FUNCTION FOR THE HOMOGENEOUS ANISOTROPIC MEDIUM

We assume a rectangular x_1, x_2, x_3 coordinate system in which E_j, H_j, or J_j represent the component of electric field, magnetic field or source current density parallel to the x_j axis, j = 1, 2, or 3. If all sources $\underset{\sim}{J}$ are oscillating according to exp(-iωt) and the ambient medium is the most general homogeneous anisotropic one [5], the reduced Maxwell's equations take the form

$$(\nabla \times \underset{\sim}{E})_j - i\omega \mu_{jn} H_n = 0 \tag{1}$$
$$(\nabla \times \underset{\sim}{H})_j + i\omega \epsilon_{jn} E_n = J_j$$

Here and subsequently the summation convention is used:

$$\mu_{jn} H_n \equiv \sum_{n=1}^{3} \mu_{jn} H_n$$

Also $\mu_{jn} \equiv \underset{\sim}{\mu}$ and $\epsilon_{jn} \equiv \underset{\sim}{\epsilon}$ are the dielectric and permeability tensors; they are 3x3 matrices or dyads with constant elements, independent of position outside of specifically designated bodies.

The Green's function, variously denoted as $\underset{\sim}{G} = \underset{\sim}{G}(\underset{\sim}{x},\underset{\sim}{x}') = G_{ij}(\underset{\sim}{x},\underset{\sim}{x}') = G_{ij}$, is a dyad or matrix connecting the radiated electric field $\underset{\sim}{E}(\underset{\sim}{x})$, $\underset{\sim}{x} = (x_1, x_2, x_3)$, with current sources $\underset{\sim}{J}(\underset{\sim}{x}')$:

$$E_i(\underset{\sim}{x}) = \iiint G_{ij}(\underset{\sim}{x},\underset{\sim}{x}')\, J_j(\underset{\sim}{x}')\, dx_1\, dx_2\, dx_3 = \int G_{ij}(\underset{\sim}{x},\underset{\sim}{x}') J_j(\underset{\sim}{x}') d^3x \tag{2}$$

To get a representation for $\underset{\sim}{G}$ we first write the reduced Maxwell's equations (1) in matrix operator form, using d_k to symbolize the operation $\partial/\partial x_k$:

$$\begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \equiv \underset{\sim}{d}\cdot\underset{\sim}{E} = i\omega\begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix}\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} \equiv i\omega\underset{\sim}{\mu}\cdot\underset{\sim}{H} \tag{3}$$

$$\underset{\sim}{d}\cdot\underset{\sim}{H} + i\omega\underset{\sim}{\epsilon}\cdot\underset{\sim}{E} = \underset{\sim}{J} \tag{4}$$

Solving the first of the foregoing for $\underset{\sim}{H}$ and substituting the result into the second, we obtain

$$\underset{\sim}{d}\cdot\underset{\sim}{\nu}\cdot\underset{\sim}{d}\cdot\underset{\sim}{E} - \omega^2\underset{\sim}{\epsilon}\cdot\underset{\sim}{E} = i\omega\underset{\sim}{J} \tag{5}$$

where $\underset{\sim}{\nu} = \underset{\sim}{\mu}^{-1}$ is the matrix inverse to $\underset{\sim}{\mu}$. Now we assume that both $\underset{\sim}{E}$ and $\underset{\sim}{J}$ have Fourier representations, i.e., are superpositions of plane waves propagating as $\exp(ik_1x_1 + ik_2x_2 + ik_3x_3) \equiv \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x})$. Thus if

$$e_n(\underset{\sim}{k}) = (2\pi)^{-3}\int E_n(\underset{\sim}{x})\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, d^3x \tag{6a}$$

then

$$E_n(\underset{\sim}{x}) = \int e_n(\underset{\sim}{k})\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, d^3k \tag{6b}$$

is the inverse relation; similar relations hold between $J_n(\underset{\sim}{x})$, $j_n(\underset{\sim}{k})$. We substitute the representations for $\underset{\sim}{E}$, $\underset{\sim}{J}$ into (5), perform the differentiations (implied in the $\underset{\sim}{d}$ matrices) under the integral signs, and equate coefficients of $e^{ik\cdot x}$ in the resulting integrands and obtain:

$$\underset{\sim}{K}\cdot\underset{\sim}{\nu}\cdot\underset{\sim}{K}\cdot\underset{\sim}{e}+\omega^2\underset{\sim}{\epsilon}\cdot\underset{\sim}{e}=-i\omega\underset{\sim}{j} \tag{7}$$

where $\underset{\sim}{K}$ represents the matrix

$$\begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix} \tag{8a}$$

so that $K_{ij} = \pm(k_1,\ k_2,\ \text{or}\ k_3)$. (8b)

Defining the matrix $\underset{\sim}{M}$ through

$$\underset{\sim}{M}\equiv\underset{\sim}{K}\cdot\underset{\sim}{\nu}\cdot\underset{\sim}{K}+\omega^2\underset{\sim}{\epsilon}$$

we have

$$\underset{\sim}{e}(\underset{\sim}{k}) = -i\omega\,\underset{\sim}{M}^{-1}\cdot j(\underset{\sim}{k}) \tag{9}$$

whence from (6b)

$$\begin{aligned}\underset{\sim}{E}(\underset{\sim}{x}) &= -i\omega\int e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\,\underset{\sim}{M}^{-1}\cdot\underset{\sim}{j}(\underset{\sim}{k})\,d^3k \\ &= -i\omega\int e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\,\underset{\sim}{M}^{-1}\cdot\int\underset{\sim}{J}(\underset{\sim}{x}')\,e^{-i\underset{\sim}{k}\cdot\underset{\sim}{x}'}(2\pi)^{-3}d^3x\,d^3k \\ &= \int\left\{-\frac{i\omega}{(2\pi)^3}\int e^{i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}\underset{\sim}{M}^{-1}d^3k\right\}\underset{\sim}{J}(\underset{\sim}{x})\,d^3x'\end{aligned} \tag{10}$$

Thus we have, from (2),

$$G_{ij}(\underset{\sim}{x},\underset{\sim}{x}') = -\frac{i\omega}{(2\pi)^3}\int e^{i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}(M^{-1})_{ij}\,d^3k = G_{ij}(\underset{\sim}{x}-\underset{\sim}{x}') \tag{11}$$

Thus the properties of $G_{ij}(\underset{\sim}{x},\underset{\sim}{x}')$ are determined by the $\underset{\sim}{k}$-dependent matrix $\underset{\sim}{M} = \underset{\sim}{K}\cdot\underset{\sim}{\nu}\cdot\underset{\sim}{K} + \omega^2\underset{\sim}{\epsilon}$, or $M_{ij} = K_{ir}\nu_{rs}K_{sj} + \omega^2\epsilon_{ij}$. From (8b) it is seen that M_{ij} is an even function of $\underset{\sim}{k}$, i.e. has the same value when $\underset{\sim}{k}$ (or k_1, k_2, k_3) is replaced by $-\underset{\sim}{k}$ (or by $-k_1, -k_2, -k_3$). The matrix element $(M^{-1})_{ij}$ is in the usual fashion the cofactor of the matrix element M_{ji} divided by the determinant $|\underset{\sim}{M}|$, and, is with M_{ij} even in $\underset{\sim}{k}$; so then are $|\underset{\sim}{M}|$ and $(\underset{\sim}{M}^{-1})_{ij}$. In the exponential of the integrand of the representation (11) one sees that $(-\underset{\sim}{k})\cdot(\underset{\sim}{x}-\underset{\sim}{x}') = \underset{\sim}{k}\cdot(\underset{\sim}{x}'-\underset{\sim}{x})$; hence replacing $\underset{\sim}{k}$ with $-\underset{\sim}{k}$ gives the representation for $G_{ij}(\underset{\sim}{x}'-\underset{\sim}{x})$ identical with that written for $G_{ij}(\underset{\sim}{x}-\underset{\sim}{x}')$:

$$G_{ij}(\underset{\sim}{x}-\underset{\sim}{x}') \equiv G_{ij}(\underset{\sim}{x}'-\underset{\sim}{x}) \tag{12}$$

Next, by examining the expansion of $K\cdot v\cdot K$, with the aid of (8a) and (8b), one sees that the cofactor of M_{ij} is of no higher than fourth degree in k_1, k_2, k_3. Finally the only possible terms of sixth degree in $|M|$ arise from $|\underset{\sim}{K}\cdot\underset{\sim}{v}\cdot\underset{\sim}{K}| = |\underset{\sim}{K}||\underset{\sim}{v}||\underset{\sim}{K}|$, but $|\underset{\sim}{K}| \equiv 0$, so that $|\underset{\sim}{M}|$ is at most of fourth degree in k_1, k_2, k_3.

Thus we have the dyadic Green's function Fourier-represented in the form

$$G_{ij}(\underset{\sim}{x}-\underset{\sim}{x}') = G_{ij}(\underset{\sim}{x}'-\underset{\sim}{x}) = -\frac{i\omega}{(2\pi)^3}\int e^{i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}\frac{P_{ij}(\underset{\sim}{k})}{Q(\underset{\sim}{k})}\,d^3k \tag{13}$$

where P_{ij} and Q are polynomials of even, at most fourth, degree in k_1, k_2, k_3. By transforming to spherical coordinates in k-space according, say, to

$$\begin{aligned} k_1 &= R\sin\theta\cos\phi \\ k_2 &= R\sin\theta\sin\phi \\ k_3 &= R\cos\theta \end{aligned} \tag{14}$$

we express P_{ij} and Q as polynomials of at most second degree in R^2 with coefficients which are functions of θ, ϕ. The transformation also replaces

$$\int d^3k \quad \text{with} \quad \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} R^2\,dR\,\sin\theta\,d\theta\,d\phi.$$

Thus the integrand is an even function of R except for the $\exp[i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')]$ factor. But, according to (12), we may replace $\underset{\sim}{x}-\underset{\sim}{x}'$ with $\underset{\sim}{x}'-\underset{\sim}{x}$, without changing the result:

$$G_{ij}(\underset{\sim}{x}-\underset{\sim}{x}') = \frac{1}{2}[G_{ij}(\underset{\sim}{x}-\underset{\sim}{x}') + G_{ij}(\underset{\sim}{x}'-\underset{\sim}{x})]$$

$$= -\frac{i\omega}{16\pi^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}[e^{ik\cdot(\underset{\sim}{x}-\underset{\sim}{x}')} + e^{-i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}]\frac{P_{ij}}{Q}R^2 dR \sin\theta \, d\theta \, d\phi$$

Now the entire integrand is even in R, so that the integral over the positive real R axis can be extended to the entire real R axis:

$$G_{ij}(\underset{\sim}{x}-\underset{\sim}{x}') = -\frac{i\omega}{16\pi^3}\int_0^{2\pi}\int_0^{\pi}\int_{-\infty}^{\infty} e^{i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}\frac{P_{ij}}{Q}R^2 dR \sin\theta \, d\theta \, d\phi \tag{15}$$

The importance of this extension is that it permits evaluating the R integration by residues, in certain applications to be discussed.

We now inquire whether $\underset{\sim}{G}$ can be obtained in closed form, i.e. whether (13) or (15) can be evaluated for the general case. When both $\underset{\sim}{\epsilon} = \epsilon_0 \underset{\sim}{I}$, $\underset{\sim}{\mu} = \mu_0 \underset{\sim}{I}$, ($\underset{\sim}{I}$ the identity matrix) and thus are essentially scalars as in the free-space case, P_{ij} and Q have common factors, cancellation of which leaves P_{ij}/Q in the form of a quotient of second degree polynomials in k_1, k_2, k_3, or first degree polynomials in R^2. Further, it is seen that typical terms $ak_1k_2k_3^2$, bk_1^2, ck_1k_2, d in the polynomial P_{ij} in the integrand of (13) are obtainable by applying the differential operators $a\frac{\partial^4}{\partial x_1 \partial x_2 \partial^2 x_3}$, $b\frac{\partial^2}{\partial x_1^2}$, $c\frac{\partial^2}{\partial x_1 \partial x_2}$, d externally to the (scalar) integral

$$g(\underset{\sim}{x}-\underset{\sim}{x}') \equiv -\frac{i\omega}{(2\pi)^3}\int e^{i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}\frac{d^3k}{Q(\underset{\sim}{k})} \tag{16a}$$

$$= -\frac{i\omega}{16\pi^3}\int_0^{2\pi}\int_0^{\pi}\int_{-\infty}^{\infty} e^{i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}')}Q^{-1}R^2 dR \sin\theta \, d\theta \, d\phi$$

Hence we have

$$G_{ij}(\underset{\sim}{x}-\underset{\sim}{x}') = O_{ij}\, g(\underset{\sim}{x}-\underset{\sim}{x}') \tag{16b}$$

where O_{ij} is a differential operator of even degree operating on $g(\underset{\sim}{x}-\underset{\sim}{x}')$ as function of $\underset{\sim}{x}$. The evaluation of the single (scalar) integral $g(\underset{\sim}{x}-\underset{\sim}{x}')$ thus suffices for obtaining $\underset{\sim}{G}$ in closed form. Here we note that if $Q(\underset{\sim}{k})$ were left as a fourth degree polynomial in $\underset{\sim}{k}$, or second degree in R^2, there is no question that the R-integral is performable by residues at the outset. (One sets $\mathrm{Im}(\epsilon_0) > 0$ in the scalar case to determine location of the zeros of $Q(\underset{\sim}{k})$; analogous methods work in the general case.) In (15), $R^2 P_{ij}$ was of higher degree in R than $Q(\underset{\sim}{k})$ so that the immediate residue-evaluation of the R-integral was not possible there.

Further discussion of the evaluation of $\underset{\sim}{G}$ (or g) is relegated to Appendix I as not germane to the main line of argument. It is shown heuristically there that $\underset{\sim}{G}$ cannot have a closed-form evaluation in this general case.

We now consider $G_{ij}(\underset{\sim}{x}-\underset{\sim}{x}') = G_{ij}(\underset{\sim}{x}'-\underset{\sim}{x}) = G_{ij}$ as a matrix, or dyad. We write

$$G_{ij} = \frac{1}{2}(G_{ij} + G_{ji}) + \frac{1}{2}(G_{ij} - G_{ji}) \equiv S_{ij} + A_{ij}$$

where $S_{ij} = S_{ji}$ is a symmetric 3x3 matrix $A_{ij} = -A_{ji}$ is an antisymmetric one. Writing $\underset{\sim}{G}^T$ as the transpose of $\underset{\sim}{G}$ we have $G_{ij} = (\underset{\sim}{G}^T)_{ji} = S_{ij} + A_{ij} = (\underset{\sim}{S}^T)_{ij} - (\underset{\sim}{A}^T)_{ij}$ or $\underset{\sim}{A}^T = -\underset{\sim}{A}$, $\underset{\sim}{S}^T = \underset{\sim}{S}$. From (8) we have $\underset{\sim}{K}^T = -\underset{\sim}{K}$ so that

$$\underset{\sim}{M}^T = (\underset{\sim}{K}\cdot\underset{\sim}{\nu}\cdot\underset{\sim}{K} + \omega^2\underset{\sim}{\epsilon})^T = (\underset{\sim}{K}\cdot\underset{\sim}{\nu}\cdot\underset{\sim}{K})^T + \omega^2\underset{\sim}{\epsilon}^T$$
$$= \underset{\sim}{K}^T\cdot\underset{\sim}{\nu}^T\cdot\underset{\sim}{K}^T + \omega^2\underset{\sim}{\epsilon}^T = \underset{\sim}{K}\cdot\underset{\sim}{\nu}^T\cdot\underset{\sim}{K} + \omega^2\underset{\sim}{\epsilon}^T$$

Thus $\underset{\sim}{M}^T = \underset{\sim}{M}$ for all $\underset{\sim}{k}$ if and only if $\underset{\sim}{\nu}^T = \underset{\sim}{\nu}$, $\underset{\sim}{\epsilon}^T = \underset{\sim}{\epsilon}$, that is, if both $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\nu}$ are symmetric matrices. Since $(\underset{\sim}{M}^{-1})^T = \underset{\sim}{M}^{-1}$, $[\underset{\sim}{\nu}^T = \underset{\sim}{\nu}]$ if and only if $\underset{\sim}{M}^T = \underset{\sim}{M}$, $[\underset{\sim}{\mu}^T = \underset{\sim}{\mu}]$ we have via P_{ij} in (13) or (15) that $\underset{\sim}{G}^T = \underset{\sim}{G}$, or $G_{ij} = G_{ji}$ only when

$\underset{\sim}{\mu}^T = \underset{\sim}{\mu}$ and $\underset{\sim}{\epsilon}^T = \underset{\sim}{\epsilon}$; i.e., when and only when both $\underset{\sim}{\epsilon}$ and $\underset{\sim}{\mu}$ are symmetric tensors do we have a symmetric Green's function $G_{ij} = G_{ji} = S_{ij}$.

In summary, we conclude that the dyadic Green's function $G_{ij}(\underset{\sim}{x})$ for electromagnetic waves in a homogeneous, anisotropic medium is an even function of its argument $\underset{\sim}{x}$. As a matrix or dyad, $\underset{\sim}{G}$ is symmetric ($G_{ij} = G_{ji}$) if and only if both dielectric and permeability tensor are symmetric. In the general case $\underset{\sim}{G}$ is representable as the sum of a symmetric and an antisymmetric dyad, each being representable through a triple Fourier representation, i.e. as a superposition of plane waves of varying propagation constants $\underset{\sim}{k}$. When the representations are given in terms of spherical R, θ, ϕ coordinates in k-space, the radial integral can be taken as one-half the integral over the full R axis. In the general case, the argument of Appendix I indicates that $\underset{\sim}{G}$ is unavailable in closed form, so that the Fourier or other representation is necessary.

3. REVIEW OF SOME ANTENNA THEORY

We apply the properties of the Green's function as summarized in the foregoing paragraph to the problem of computing the input impedance Z of a perfectly conducting antenna fed at a gap. The theory for antennas immersed in a homogeneous anisotropic medium is not discussed in the standard works, and for this reason we modify some standard antenna theory to take account of the generalization.

We take as the ideal antenna under consideration a smooth closed surface Σ which is perfectly conducting everywhere except in a slit, or antenna gap, of infinitesimal width Δ, 'cut' along a smooth curve connecting the surface points $\underline{a}$ and $\underline{b}$. Let $\underline{u}$ measure distance from $\underline{a}$ along the slit, and assume a surface current-density $\underset{\sim}{J}(u)$ flowing normal to the slit as the antenna input current distribution. The antenna input current is then defined as $I = \int_a^b J(u)du$. As a result of this input, the surface current density $\underset{\sim}{j}(\underset{\sim}{s})$ will be found at the general point $\underset{\sim}{s}$ of Σ, with $\underset{\sim}{j}(\underset{\sim}{s})$ defined as $\underset{\sim}{J}(u)$ when $\underset{\sim}{s}$ lies in the slit at distance $\underline{u}$ from $\underline{a}$.

The surface current density $\underset{\sim}{j}(\underset{\sim}{s})$ radiates, so that the electric field $\underset{\sim}{E}(\underset{\sim}{x})$ at a general point $\underset{\sim}{x}$ is given through the Green's function:

$$\underset{\sim}{E}(\underset{\sim}{x}) = \int_\Sigma \underset{\sim}{G}(\underset{\sim}{x} - \underset{\sim}{s}') \cdot \underset{\sim}{j}(\underset{\sim}{s}')\, d^2s' \tag{17}$$

in which the conversion from the volume integral (2) to the surface integral (17) is assumed properly made. Letting $\underset{\sim}{x}$ be at the surface point $\underset{\sim}{s}$, and taking account of the fact that the tangential component $\underset{\sim}{E}_t(\underset{\sim}{s})$ of the electric field must vanish at each surface point $\underset{\sim}{s}$ of a perfect conductor, we have

$$\underset{\sim}{E}_t(\underset{\sim}{s}) = \int_\Sigma \underset{\sim}{G}(\underset{\sim}{s} - \underset{\sim}{s}') \cdot \underset{\sim}{j}(\underset{\sim}{s}')\, d^2s' = \begin{cases} E_t(u), \text{ say, at position u in the slit} \\ 0, \text{ elsewhere on } \Sigma. \end{cases} \tag{18}$$

where the subscript t denotes the component tangent to Σ at $\underset{\sim}{s}$. Now we define a mean antenna voltage V through

$$IV = V\int_a^b J(u)\,du = \Delta \int_a^b du\, \underset{\sim}{E}_T(u) \cdot \underset{\sim}{J}(u) \tag{19}$$

and, finally, define an antenna impedance Z through V = ZI. We now have

$$\begin{aligned} I^2 Z = VI &= \Delta \int_a^b \underset{\sim}{J}(u) \cdot \int_\Sigma \underset{\sim}{G}[\underset{\sim}{s}(u) - \underset{\sim}{s}'] \cdot \underset{\sim}{j}(\underset{\sim}{s}')\, d^2s'\, du \\ &= \int_\Sigma \int_\Sigma \underset{\sim}{j}(\underset{\sim}{s}) \cdot \underset{\sim}{G}(\underset{\sim}{s} - \underset{\sim}{s}') \cdot \underset{\sim}{j}(\underset{\sim}{s}')\, d^2s\, d^2s' \end{aligned} \tag{20}$$

where, in the last substitution, advantage was taken of eq. (18) and the definition of $\underset{\sim}{j}(\underset{\sim}{s})$ in the slit. Now

$$\begin{aligned} I^2 Z &= \int_\Sigma \int_\Sigma \underset{\sim}{j}(\underset{\sim}{s}) \cdot \underset{\sim}{G}(\underset{\sim}{s} - \underset{\sim}{s}') \cdot \underset{\sim}{j}(\underset{\sim}{s}')\, d^2s\, d^2s' \\ &= \int_\Sigma \int_\Sigma \underset{\sim}{j}(\underset{\sim}{s}') \cdot \underset{\sim}{G}(\underset{\sim}{s}' - \underset{\sim}{s}) \cdot \underset{\sim}{j}(\underset{\sim}{s})\, d^2s\, d^2s' \end{aligned} \tag{21a}$$

which by virtue of (12), is

$$I^2 Z = \iint_{\Sigma\,\Sigma} \underset{\sim}{j}(\underset{\sim}{s}') \cdot \underset{\sim}{G}(\underset{\sim}{s}-\underset{\sim}{s}') \cdot \underset{\sim}{j}(\underset{\sim}{s})\, d^2s\, d^2s' \tag{21b}$$

Writing the matrix $\underset{\sim}{G}$ as the sum of a symmetric matrix $\underset{\sim}{S}$ and an antisymmetric matrix $\underset{\sim}{A}$, we have

$$\underset{\sim}{G} = \underset{\sim}{S} + \underset{\sim}{A},$$

where

$$\underset{\sim}{j} \cdot \underset{\sim}{S} \cdot \underset{\sim}{k} = \underset{\sim}{k} \cdot \underset{\sim}{S} \cdot \underset{\sim}{j}\,; \quad \underset{\sim}{j} \cdot \underset{\sim}{A} \cdot \underset{\sim}{k} = -\underset{\sim}{k} \cdot \underset{\sim}{A} \cdot \underset{\sim}{j}$$

$\underset{\sim}{j}$, $\underset{\sim}{k}$ being arbitrary vectors. It is seen that $\underset{\sim}{G}$ may be replaced with $\underset{\sim}{S}$ in (21a, b), the contribution of $\underset{\sim}{A}$ to Z vanishing. Still stipulating that the input current $\underset{\sim}{J}(u)$ represents the value of the surface current $\underset{\sim}{j}(\underset{\sim}{s})$ for $\underset{\sim}{s}$ in the slit, we have a variational estimate for Z in

$$I^2 Z = \int_\Sigma \int_\Sigma \underset{\sim}{j}(\underset{\sim}{s}) \cdot \underset{\sim}{S}(\underset{\sim}{s}-\underset{\sim}{s}') \cdot \underset{\sim}{j}(\underset{\sim}{s}')\, d^2s\, d^2s' \tag{22}$$

provided that one can replace $\underset{\sim}{G}$ with $\underset{\sim}{S}$ in (18). This replacement is obviously correct when both the $\underset{\sim}{\epsilon}$ and the $\underset{\sim}{\mu}$ tensors are symmetric, for then $\underset{\sim}{A} \equiv 0$ and $\underset{\sim}{G} \equiv \underset{\sim}{S}$. (The replacement is also justified under certain other circumstances, to be detailed later.) For, suppose we replace the exact $\underset{\sim}{j}(\underset{\sim}{s})$ with the approximate $\underset{\sim}{j}'(\underset{\sim}{s}) = \underset{\sim}{j}(\underset{\sim}{s}) + \alpha\underset{\sim}{k}(\underset{\sim}{s})$ where $\underset{\sim}{k}(\underset{\sim}{s})$ is an arbitrary surface current distribution vanishing in the slit, and α is a parameter whose 'best' value is to be determined. Writing

$$I^2 Z' = \iint_{\Sigma\,\Sigma} \underset{\sim}{j}'(s) \cdot \underset{\sim}{S}(\underset{\sim}{s}-s') \cdot j'(s')\, d^2s'\, d^2s,$$ we have

$$\begin{aligned} I^2 Z' &= \iint_{\Sigma\,\Sigma} \underset{\sim}{j}(\underset{\sim}{s}) \cdot \underset{\sim}{S}(\underset{\sim}{s}-\underset{\sim}{s}') \cdot \underset{\sim}{j}(\underset{\sim}{s}')\, d^2s\, d^2s \\ &\quad + \alpha \iint_{\Sigma\,\Sigma} [\underset{\sim}{j}(\underset{\sim}{s}) \cdot \underset{\sim}{S}(\underset{\sim}{s}-\underset{\sim}{s}') \underset{\sim}{k}(\underset{\sim}{s}') + \underset{\sim}{k}(\underset{\sim}{s}) \cdot \underset{\sim}{S}(\underset{\sim}{s}-\underset{\sim}{s}') \cdot \underset{\sim}{j}(\underset{\sim}{s}')]\, d^2s\, d^2s' + O(\alpha^2) \\ &= I^2 Z + O(\alpha^2) \end{aligned} \tag{23}$$

since the coefficient of $\underset{\sim}{k}(\underset{\sim}{s})$ or $\underset{\sim}{k}(\underset{\sim}{s}')$ vanishes, by (18), except in the slit, where $\underset{\sim}{k}$ vanishes by assumption. We therefore have both $Z' = Z$ at $\sigma = 0$ and $dZ'/d\sigma = 0$ at $\sigma = 0$. Thus if the guessed antenna current distribution $\underset{\sim}{j}'(\underset{\sim}{s})$ is a good approximation (as judged by the smallness of σ, when the 'error current' satisfies say $\int_{\Sigma} \underset{\sim}{k} \cdot \underset{\sim}{k} d^2 s = 1$), the estimate Z' is even closer to Z, by $O(\sigma^2)$, as compared with $O(\sigma)$ for the error in the assumed current. This means that we may take a parametrized trial current distribution, regarded as accurate for some particular 'best' set of parameter values, and choose as 'best' that set for which each parameter-derivative of Z' as computed in (23) vanishes. The computed Z' is then to be regarded as a better approximation to Z than the 'best' $\underset{\sim}{j}'$ is to $\underset{\sim}{j}$.

We are not entirely sure that the Z as computed above is in fact the antenna impedance in any practical case; details of the gap geometry and currents have been brushed over here, but relevant considerations are discussed qualitatively in Appendix II.

For the case that $\underset{\sim}{G}$ is non-symmetric, so that $\underset{\sim}{A} \neq 0$, the formula (22) for Z retains its variational character provided that the antenna is constructed of Faraday screen material such that all conducting elements are parallel to some fixed vector $\underset{\sim}{v}$. For then, in eqs. (18)-(22) the vectors $\underset{\sim}{E}_t(\underset{\sim}{s})$ and $\underset{\sim}{j}(\underset{\sim}{s}')$ have the form $e(\underset{\sim}{s})\underset{\sim}{v}$, $j(\underset{\sim}{s}')\underset{\sim}{v}$; the tangential field contribution $e_a(\underset{\sim}{s})\underset{\sim}{v}d^2 s'$ owing to the current $\underset{\sim}{j}(\underset{\sim}{s}')d^2 s' = j(\underset{\sim}{s}')\underset{\sim}{v}d^2 s'$, arising from the antisymmetric term $\underset{\sim}{A}$ of $\underset{\sim}{G}$, has amplitude given by $e_a(\underset{\sim}{s})(\underset{\sim}{v} \cdot \underset{\sim}{v})d^2 s' = \{\underset{\sim}{v} \cdot \underset{\sim}{A}(\underset{\sim}{s} - \underset{\sim}{s}') \cdot \underset{\sim}{v}\} (j(\underset{\sim}{s}')d^2 s')$. But $\{\underset{\sim}{v} \cdot \underset{\sim}{A} \cdot \underset{\sim}{v}\} \equiv 0$, owing to the antisymmetry of $\underset{\sim}{A}$ as a matrix of dyad. Thus the electric field tangential to the Faraday-screen elements is strictly determined by the symmetric part of the Green's function, regardless of the current distribution assumed flowing in the elements or the antisymmetry of the ambient medium.

Insofar as an antenna built of thin, mutually parallel conductors can be regarded as approximated by a similar one built of parallel Faraday screen material, one can thus compute the impedance for the antenna with a variational form.

The "complex power" formulation of the impedance is obtained as follows: With $V = ZI$ as before, and letting I^*, j^* denote the complex conjugate of $I, \underset{\sim}{j}$ we have, following the general logic of eqs. (17) through (20),

$$Z(II^*) = Z|I^2| = I^* V = \iint_{\Sigma\,\Sigma} \underset{\sim}{j}^*(\underset{\sim}{s}) \cdot \underset{\sim}{G}(\underset{\sim}{s} - \underset{\sim}{s}') \cdot \underset{\sim}{j}(\underset{\sim}{s}')\, d_s^2\, d_{s'}^2 \tag{24}$$

The double surface integral on the right differs from that of the "variational" form, eq. (21b), only through the replacement of one of the surface currents $\underset{\sim}{j}$ by its complex conjugate. But this replacement is enough to destroy the variational character so that the form (24) does not permit estimating "best" parameters when Z is computed through a parametrized assumed current distribution; also the Z-estimate for a given current distribution is better when made by (21b) than by (24), again owing to the variational character of the former. When the antenna geometry and asymmetry of the medium are such that (21b) is non-variational it is conjectured that (21b) or (22) is still preferable to (24), but we have no means of proving this. On the other hand the two formulations pose about the same mathematical problem in their evaluation, and give the same impedance-values when correct antenna current distributions are used.

It is clear that the present argument can be used to show that Schwinger's variational expression, for the scattering by a perfectly conducting object, will retain its variational quality for the anisotropic ambient medium when all currents, including those representing transmitting and receiving dipoles, can be assumed mutually parallel. In general, variational expressions and reciprocity, i.e. symmetry on the part of the Green's function, appear to be closely intertwined concepts, and one conjectures that a variational expression for a quantity of interest (impedance, scattering amplitude) is obtainable only when any antisymmetry in the Green's function has no effect in the quantity considered.

4. APPLICATION TO THE THIN CYLINDRICAL DIPOLE IMPEDANCE CALCULATION

We have reported elsewhere [1] numerical calculations of the impedance of a thin center-fed hollow cylindrical dipole antenna of radius r, half-length L in the homogeneous magnetosphere. The ambient medium was assumed to have the scalar permeability of vacuum and to be anisotropic through a uniaxial non-symmetric dielectric tensor. Taking an R,ϕ,z cylindrical coordinate system with z measuring distance along the dipole's axis from its center, $\hat{\underset{\sim}{z}}$ the unit vector parallel to the z-axis, we assumed

(a) that the current-density on the dipole's surface is independent of azimuth, and

(b) that the current elements are everywhere parallel to the z-axis.

Assumption (b) is the same as assuming the dipole made of Faraday-screen material with elements everywhere parallel to the axis; azimuthal current components, clearly possible with thick dipoles in a gyratory medium, are thus assumed suppressed, or at least negligible as far as the impedance is concerned.

Thus the "complex power" calculation takes the form

$$Z(2\pi r)^2 j(0)\, j^*(0) =$$

$$N\int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \int_{-L}^{L} dz \int_{-L}^{L} dz'\, j^*(z)\, j(z')\, \hat{\underset{\sim}{z}} \cdot \underset{\sim}{G}[\underset{\sim}{x}(r,\phi,z) - \underset{\sim}{x}(r,\phi',z')] \cdot \hat{\underset{\sim}{z}} \qquad (25)$$

where $j(0)$ gives the current density across the central circumferential antenna gap, and where the constant N contains numerical and dimensional constants unessential to the following discussion. One obtains the variational form by simply removing the asterisks. For the isotropic 'vacuum' case we may use the known closed form for $\underset{\sim}{G}$ here, giving a fourfold integral, or double surface integral, to be evaluated for estimating Z from an assumed $j(z)$. For our gyratory medium we are forced to a representation for $\underset{\sim}{G}$ and find it convenient to use the Fourier representation (13) for discussion. (In the actual calculation we used (15).) This gives (25) in the form of a seven-fold integral from which we select out only the double integral over the azimuth angles ϕ, ϕ' for further discussion. In the $\exp(i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{x}'))$ of (13) we choose the x_3 axis identical with the dipole's, or the z-axis, so that one writes conveniently $x_1 = R\cos\phi$, $x_2 = R\sin\phi$. Under assumptions (a) and (b), ϕ and ϕ' now appear in (25) only in the exponential of the Green's function representation, and, with $R = r$ on the dipole's surface in the form

$$\exp[irk_1(\cos\phi - \cos\phi') + ik_2 r(\sin\phi - \sin\phi')] \equiv E \tag{26}$$

After dividing out the common factor r^2, eq. (25) now has the form

$$\begin{aligned} Z = Z(r) &= \left\{ \frac{N}{j(0)\, j^*(0)} \int d^3k \int_{-L}^{L} dz \int_{-L}^{L} dz'\, j^*(z)\, j(z')\, e^{ik_3(z-z')} \right\} \frac{1}{4\pi^2} \int_0^{2\pi}\!\!\int_0^{2\pi} E\, d\phi\, d\phi' \\ &= \{\ \} \frac{1}{4\pi^2} \int_0^{2\pi}\!\!\int_0^{2\pi} e^{ir[k_1(\cos\phi - \cos\phi') + k_2(\sin\phi - \sin\phi')]}\, d\phi\, d\phi' \\ &= \{\ \}\, J_0^2\left(r\sqrt{k_1^2 + k_2^2}\right) \end{aligned} \tag{27}$$

We are now in a position to discuss errors entailed in the frequently made assumption

(c) that the radiating current $j(z')$, assumed in (25) and (27) to be distributed over the dipole's surface, can be replaced by a current of the same magnitude and z-dependence flowing along the dipole's axis.

On this assumption and in the notation of (27) we would therefore calculate an approximate Z_c according to

$$Z_c = Z_c(r) = \{\,\}\,\frac{1}{4\pi^2}\int_0^{2\pi}\!\!\int_0^{2\pi} e^{ir(k_1\cos\phi + k_2\sin\phi)}\,d\phi\,d\phi' \tag{28}$$

From (27) we now have

$$\begin{aligned} Z(r) &= \{\,\}\,J_0^2\left(r\sqrt{k_1^2+k_2^2}\right) = \{\,\}\,\frac{1}{\pi}\int_0^{\pi} J_0\left[(2r\sin\theta)\sqrt{k_1^2+k_2^2}\right]d\theta \\ &= \{\,\}\,\frac{1}{2\pi^2}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, e^{i(2r\sin\theta)(k_1\cos\phi + k_2\cos\phi)} \\ &= \frac{1}{\pi}\int_0^{\pi} Z_c(2r\sin\theta)\,d\theta \end{aligned} \tag{29}$$

in which J_0 is the Bessel function and the integrals are formed in Watson's "Bessel Functions", pp. 31-32. Thus the 'exact' Z is found from the approximate Z_c calculated under assumption (c); the same relation (29) will hold for the variational calculation as well as for the complex power form, since the former is obtained from the latter on eliminating the asterisks in the $\{\ \}$ of (27). Similarly, the limits $\pm$ L of the z,z' integrals in $\{\ \}$, specific to the center-fed dipole, will differ for other thin-cylinder impedance problems, without affecting the applicability of (29).

For the thin dipole the radius-to-half-length ratio r/L is small, and it is appropriate to expand Z or Z_c in powers of this parameter. Actually it is found that

$$Z_c(r) = \log\left(\frac{r}{L}\right)\sum_{n=0}^{\infty} A_n\left(\frac{r}{L}\right)^n + \sum_{n=0}^{\infty} B_n\left(\frac{r}{L}\right)^n \tag{30}$$

where A_n and B_n are independent of r but depend on the form assumed for j(z), on L, on the dipole's orientation and on the properties of the medium at angular frequency ω. The correction (29), eliminating any errors arising from assumption (c), is then

$$
\begin{aligned}
Z &= \sum_{0}^{\infty} A_n \int_0^{\pi} \log[(2r\sin\theta/L)][(2r\sin\theta/L)]^n \frac{d\theta}{\pi} \\
&\quad + \sum_{0}^{\infty} B_n \int_0^{\pi} [(2r\sin\theta/L)]^n \frac{d\theta}{\pi} \\
&= A_0 \log\left(\frac{r}{L}\right) + B_0 + O[(r/L)\log(r/L)]
\end{aligned}
\tag{31}
$$

Thus Z_c and Z are identical in their two leading terms, which are dominant for $r/L \ll 1$; higher order corrections can be made as necessary by known integrals.

The prescription (29) for obtaining Z from Z_c is the result of being driven to use a Fourier representation for $\underset{\sim}{G}$ rather than the closed form of the usual calculations; existing discussion [6] of the error $Z-Z_c$ based on the closed-form $\underset{\sim}{G}$ of the isotropic case appears to be relatively qualitative.

APPENDIX I

PROSPECTS OF OBTAINING A CLOSED-FORM DYADIC GREEN'S FUNCTION

We investigate the prospects of obtaining in closed form a dyadic Green's function for the reduced Maxwell's equations for a homogeneous anisotropic medium. We are specifically interested in a medium having a uniaxial tensor dielectric such as that pertaining to the magnetosphere, or a uniaxial tensor permeability appropriate to a magnetized ferrite.

We have obtained, in eq. (16a) of the text, a scalar integral $g(\underset{\sim}{x})$, to which appropriate differential operators may be applied to obtain the dyadic Green's function: Essentially,

$$g(\underset{\sim}{x}) = \int e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}}\, d^3k / Q(\underset{\sim}{k}) \tag{A1}$$

where $Q(\underset{\sim}{k})$ is a polynomial in $k_1, k_2, k_3 \equiv p, q, r$ of even degree no higher than the fourth. The roots of $Q(\underset{\sim}{k}) = 0$ for a given direction $\hat{\underset{\sim}{k}}$ are four in number, corresponding to two plane waves (of differing polarizations) running in both the positive and the negative $\hat{\underset{\sim}{k}}$ direction. (This is seen more easily by examining plane-wave solutions of the reduced Maxwell's equations (1) or (5).) The integral (A1) thus represents a superposition of (scalar) plane waves having (at least nearly) common phases at the source $\underset{\sim}{x} = 0$. The asymptotic behavior of $\underset{\sim}{G}$ in the neighborhood of a point $\underset{\sim}{x}$ far distant from the source is that of two plane waves having phases determined by $\underset{\sim}{x}$ and amplitudes determined by evaluating the integral (A1) asymptotically. Eq. (16a) was proposed as a starting point here; one might do the R integration of the last form first, leaving the θ, ϕ integrals to the end to specify the amplitudes, at least for the asymptotic "far field" approximation valid for large $|\underset{\sim}{x}|$.

More quantitatively, one regards the g of (A1) as the analogue of $g = \exp(ik_0 r)/r$, $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, with k_0 giving all four roots of $Q(\underset{\sim}{k}) = 0$ [at least those not cancelled by the P_{ij} of (13), (15)] in the isotropic 'free space' case. Here the scalar Green's function is of the form $A(x,y,z)e^{i\phi(x,y,z)}$, and we look for a generalization of the form $A_1(\underset{\sim}{x})e^{i\phi_1(\underset{\sim}{x})} + A_2(\underset{\sim}{x})e^{i\phi_2(\underset{\sim}{x})}$ where the amplitudes A and the phases ϕ are given explicitly in terms of x,y,z and the constants of the medium. For large $\underset{\sim}{x}$, the amplitudes vary slowly and the main variation of g is according to the phases $\phi(x,y,z)$. Given the explicit g, we should therefore be able to determine the constant-phase surfaces $\phi(x,y,z) = \phi_0$; this takes on physical significance, and permits asymptotic mathematical methods, for large $|\underset{\sim}{x}|$, i.e for large ϕ_0, since $\phi \propto |\underset{\sim}{x}|$ for large $|\underset{\sim}{x}|$ and fixed direction $\hat{\underset{\sim}{x}}$.

The problem of finding the constant-phase surface or surfaces is that of finding the envelope of the plane-wave phases $\underset{\sim}{k}\cdot\underset{\sim}{x} = \phi_0$ subject to the condition that $Q(\underset{\sim}{k}) = 0$. This is basically identical with the problem of evaluating (A1) by stationary-phase methods after performing one integration (say the R-integration of (16a)) by residues. Alternatively, $Q(\underset{\sim}{k}) = 0$ gives the equation for propagation constants $\underset{\sim}{k}_e$, $\underset{\sim}{k}_o$ of the (two) kinds of plane waves leaving the point source with approximately common phases; at large distance $|\underset{\sim}{x}|$ their phases are given, with relative error decreasing with $|\underset{\sim}{x}|$, by $\underset{\sim}{k}_e\cdot\underset{\sim}{x}$, $\underset{\sim}{k}_o\cdot\underset{\sim}{x}$ so that the problem of finding the constant phase surface is to find the envelope surface of the system $\phi_0 = \underset{\sim}{k}\cdot\underset{\sim}{x}$, $Q(\underset{\sim}{k}) = 0$. Mathematically the problem is one of eliminating $\underset{\sim}{k} = p,q,r$ among the four equations $Q(p,q,r) = 0$, $\phi_0 = px + qy + rz$, $x/Q_p = y/Q_q = z/Q_r$, where the subscripts denote the partial derivatives. This elimination is possible only when ϕ_0,x,y,z satisfy a relation, which we prefer in the form $\phi_0 = \phi(x,y,z)$.

This envelope problem is of the type treated in 19th century works on algebraic curves. We do not find an explicit consideration of our problem in the most general case, but do find results pertaining to uniaxial geometries of the type encountered in the magnetospheric and ferrite media. In these cases, we may take the z-axis in the principal direction, the x-axis in any orthogonal direction, and ignore the y-variation by virtue of the rotational symmetry of the problem about the z-axis. For the magnetospheric case, q is set to zero in $Q(p,q,r)$ and the resulting $Q(p,r)$ is biquadratic in both p and r. The envelope problem is now one of finding the envelope of the lines $\phi_0 = px + rz$, where p and r satisfy the biquadratic relation $Q(p,r) = 0$.

This problem is treated in Salmon's "Higher Plane Curves" (Third edition, 1879) sections 90, 92, 298, and 300. According to our understanding, the equation of the envelope is given there in the equation $S^3 = 27T^2$, where, in Salmon's notation, $S = (\alpha 12)^4$ is a homogeneous polynomial of second degree in ϕ_0^2, x^2, z^2, and $T = (\alpha 12)^2(\alpha 23)^2(\alpha 31)^2$ is homogeneous and of third degree in these variables. (The notation does not imply that S and T have a common factor.) Salmon gives S and T explicitly in terms of the coefficients of Q. According to our algebra, the resulting envelope equation is homogeneous and of degree 6 in the foregoing variables, and the coefficients of ϕ_0^{12}, ϕ_0^{10} vanish only in exceptional circumstances. Thus to find ϕ_0 in terms of x,z one has generally to solve a sextic in ϕ_0^2 with roots function of x^2, z^2. No explicit formula for any root is to be obtained; furthermore, one would have to assign physical significance to all six roots.

If our understanding of the envelope problem and the implied algebraic manipulations has been correct, there appears no hope of finding an explicit closed-form Green's function by any process. For one would then have explicit formulas for constant-phase surfaces which certainly are asymptotic to wave-front envelopes of the foregoing type at great distances. This being the case, one would have developed explicit closed-form expressions for the roots of a sextic in terms of the coefficients, known to be impossible in the general case.

Thus we are restricted to a representation for the Green's function. This does not mean, however, that an alternative implicit algebraic representation might not be possible or useful, or that the foregoing interpretation and algebra is not open to question, or that special cases where the sextic reduces to lower degree should not be examined for physical interest.

APPENDIX II

REMARKS ON THE PROBLEM OF ESTIMATING THE IMPEDANCE OF AN ANTENNA IN THE MAGNETOSPHERE

Our purpose here is to discuss some aspects of the behavior of practical antennas in the magnetosphere, in order to justify our not discussing the effect of our elementary assumptions concerning the antenna gap on the calculated impedances.

The magnetosphere is a thin plasma, so that a solid antenna surface acts as a surface of ion-electron recombination; consequently, a plasma sheath is formed around the antenna with dimensions determined by the plasma's density and temperature. This sheath represents an inhomogeneity in the medium, so that an antenna impedance calculation, based on our assumption of a homogeneous medium, needs correction of presently unestimated amount, and by presently undetermined methods.

When used in transmission, the antenna will create locally high radio-frequency electric fields. The magnetospheric plasma presents a nonlinear medium for high fields, as evidenced by the Luxembourg effect, in which the transmitters are on the ground. We are only beginning to have experimental or theoretical estimates as to what effects such nonlinearities may have on antenna performance [7]. An experimental beginning here has been made by measuring the impedance and transmitted fields of an antenna in the magnetosphere as function of its input power level.

The practical antenna configuration is not that of the geometrically pure dipole considered here, but one in which the antenna proper is a structure attached to a metallic capsule containing power and telemetry equipment. The gap problem is thus obscured in practice, and it seems advisable to measure and interpret impedances observed with practical magnetospheric antennas in terms of differences between impedances observed in the magnetosphere and those observed in the lower atmosphere, or in essentially vacuum conditions at very great altitudes. Here the errors due to assumptions concerning gap geometries and currents are common and subtract out when impedance differences are considered. The same holds for the known effect of the finite capacity of the solid or round or finitely-thick end surfaces of practical dipoles; the neglect of such capacities here leads to known discrepancies between the predicted and measured impedances of ideal dipoles in the 'free space' medium, but their effect should subtract out again when impedance differences are under consideration.

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